Universality of Entanglement Creation in Low-Energy Two-Dimensional Scattering *†

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Abstract

We prove that the entanglement created in the low-energy scattering of two particles in two dimensions is given by a universal coefficient that is independent of the interaction potential. This is strikingly different from the three dimensional case, where it is proportional to the total scattering cross section. Before the collision the state is a product of two normalized Gaussians. We take the purity as the measure of the entanglement after the scattering. We give a rigorous computation, with error bound, of the leading order of the purity at low-energy. For a large class of potentials, that are not assumed to be spherically symmetric, we prove that the low-energy behaviour of the purity, $\mathcal{P}$, is universal. It is given by $\mathcal{P} = 1 - \frac{1}{(\ln(\sigma/\hbar)^2}\,\mathcal{E}$, where σ is the variance of the Gaussians and the entanglement coefficient, $\mathcal{E}$, depends only on the masses of the particles and not on the interaction potential. The entanglement depends strongly in the difference of the masses. It takes its minimum when the masses are equal, and it increases rapidly with the difference of the masses.

1 Introduction

In this paper we consider the low-energy scattering of two particles without spin in two dimensions. The interaction between the particles is given by a general potential that is not required to be spherically symmetric. Before the scattering the particles are in an incoming asymptotic state that is a product of two s. After the scattering the particles are in an outgoing asymptotic state that is not a product state. The problem that we solve is to compute the loss of purity of one of the particles, due to the entanglement with the other, that is produced by the collision.

In the configuration representation the Hilbert space of states for the two particles is $\mathcal{H} := L^2(\mathbb{R}^6)$. The dynamics of the particles is given by the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\varphi(\mathbf{x}_1,\mathbf{x}_2) = H\varphi(\mathbf{x}_1.\mathbf{x}_2), \tag{1.1}$$

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where the Hamiltonian is the following operator,

$$H = H_0 + V(\mathbf{x}_1 - \mathbf{x}_2), \tag{1.2}$$

where H_0 is the free Hamiltonian,

$$H_0 := -\frac{\hbar^2}{2m_1}\Delta_1 - \frac{\hbar^2}{2m_2}\Delta_2. \tag{1.3}$$

By $\hbar$ it is denoted Planck's constant, $m_j, j = 1, 2$, are, respectively, the mass of particle one and two, and Δ_j, is the Laplacian in the coordinates $\mathbf{x}_j, j = 1, 2$, of particle one and two. The potential of interaction is multiplication by a real-valued function, $V(\mathbf{x})$, defined for $\mathbf{x} \in \mathbb{R}^3$. We suppose that the interaction depends on the difference of the coordinates $\mathbf{x}_1 - \mathbf{x}_2$, but we do not require the spherical symmetry of the potential. We assume that V satisfies mild assumptions on its regularity and its decay at infinity, namely that.

ASSUMPTION 1.1.

$$(1 + |\mathbf{x}|)^\beta V(\mathbf{x}) \in L^2(\mathbb{R}^2), \quad \text{for some}\, \beta > 11. \tag{1.4}$$

Under this assumption H is a selfadjoint operator.

We also suppose that at zero energy there is neither an eigenvalue nor a resonance (half-bound state), for the Hamiltonian for the relative motion $H_{\text{rel}} := -\frac{\hbar^2}{2m} + V(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$ is the relative distance and m is the reduced mass. A zero energy resonance (half-bound state) is a bounded solution to $H_{\text{rel}}\varphi = 0$ that is not in $L^2(\mathbb{R}^2)$. See [1] for a precise definition. For generic potentials V there is neither a resonance nor an eigenvalue at zero for H_{rel}. That is to say, if we consider the potential λV with a coupling constant λ, zero can be a resonance and/or an eigenvalue for at most a finite or denumerable set of λ's without any finite accumulation point.

We study our problem in the center-of-mass frame. We consider an incoming asymptotic state that is a product of two normalized Gaussian, given in the momentum representation by,

$$\varphi_{\text{in},\mathbf{p}_0}(\mathbf{p}_1, \mathbf{p}_2) := \varphi_{\mathbf{p}_0}(\mathbf{p}_1)\, \varphi_{-\mathbf{p}_0}(\mathbf{p}_2), \tag{1.5}$$

with,

$$\varphi_{\mathbf{p}_0}(\mathbf{p}_1) := \frac{1}{(\sigma^2\pi)^{1/2}} e^{-(\mathbf{p}_1 - \mathbf{p}_0)^2/2\sigma^2}, \tag{1.6}$$

where $\mathbf{p}_i, i = 1, 2$ are, respectively, the momentum of particles one and two.

In our incoming asymptotic state (1.5) particle one has mean momentum $\mathbf{p}_0$ and particle two has mean momentum $-\mathbf{p}_0$. Both particles have the same variance, σ, of the momentum distribution. As we suppose that the scattering takes place at the origin at time zero, the average position of both particles is zero in the incoming asymptotic state (1.5). The outgoing asymptotic state of the two particles, $\varphi_{\text{out},\mathbf{p}_0}$, after the scattering process, is given by,

$$\varphi_{\text{out},\mathbf{p}_0}(\mathbf{p}_1, \mathbf{p}_2) := \left(\mathcal{S}(\mathbf{p}^2/2m)\varphi_{in,\mathbf{p}_0}\right)(\mathbf{p}_1, \mathbf{p}_2). \tag{1.7}$$

Here, $\mathbf{p} := \frac{m_2}{m_1+m_2}\mathbf{p}_1 - \frac{m_1}{m_1+m_2}\mathbf{p}_2$ is the relative momentum, $m := m_1\, m_2/(m_1+m_2)$ is the reduced mass, and $\mathcal{S}(\mathbf{p}^2/2m)$ is the scattering matrix for the relative motion. Observe that in the state (1.5) the mean relative momentum of the particles is equal to $\mathbf{p}_0$.

The measure of entanglement of a pure-bipartite state that we use is the purity of one of the particles. Namely, the trace of the square of the reduced density matrix of one of the particles, that is obtained by taking the trace on the other particle of the density matrix of the pure state. Note that the purity of a product state is one.

The purity of $\varphi_{\text{out},\mathbf{p}_0}$ is given by,

$$\mathcal{P}(\varphi_{\text{out},\mathbf{p}_0}) = \int_{\mathbb{R}^{12}} d\mathbf{p}_1\, d\mathbf{p}_1'\, d\mathbf{p}_2\, d\mathbf{p}_2'\, \varphi_{\text{out},\mathbf{p}_0}(\mathbf{p}_1,\mathbf{p}_2)\, \overline{\varphi_{\text{out},\mathbf{p}_0}(\mathbf{p}_1',\mathbf{p}_2)}\, \varphi_{\text{out},\mathbf{p}_0}(\mathbf{p}_1',\mathbf{p}_2')\, \overline{\varphi_{\text{out},\mathbf{p}_0}(\mathbf{p}_1,\mathbf{p}_2')}. \tag{1.8}$$

Since the relative momentum, $\mathbf{p}$, depends on $\mathbf{p}_1$ and on $\mathbf{p}_2$, $\varphi_{\text{out},\mathbf{p}_0}$ is no longer a product state and it has purity smaller than one. This implies that the collision has created entanglement between the two particles.

To be in the low-energy regime the following two conditions have to be satisfied. 1. The mean relative momentum $\mathbf{p}_0$ has to be small. 2. The variance σ has to be small. Note that if σ is large the incoming asymptotic state $\varphi_{\text{in},\mathbf{p}_0}$ has a big probability of having large momentum, even if the mean relative momentum $\mathbf{p}_0$ is small.

Let us designate by φ_{in} the incoming asymptotic state with mean relative momentum $\mathbf{p}_0 = 0$. The corresponding outgoing asymptotic state is $\varphi_{\text{out}} := \mathcal{S}(\mathbf{p}^2/2m)\varphi_{\text{in}}$.

We denote by

$$\mu_i := \frac{m_i}{m_1+m_2}, i = 1, 2, \tag{1.9}$$

the fraction of the mass of the i particle to the total mass.

In Theorems 3.3 and 3.6 in Section 3 we give a rigorous proof of the following results on the leading order of the purity at low energy.

$$\mathcal{P}(\varphi_{\text{out},\mathbf{p}_0}) = \mathcal{P}(\varphi_{\text{out}}) + \frac{|\mathbf{p}_0|}{\sigma}\, O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right), \qquad \text{as}\, \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0, \tag{1.10}$$

$$\mathcal{P}(\varphi_{\text{out}}) = 1 - \frac{1}{(\ln(\sigma/\hbar))^2}\mathcal{E}(\mu_1) + O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right), \qquad \text{as}\, \sigma/\hbar \to 0, \tag{1.11}$$

where $\mathcal{E}(\mu_1)$ is the entanglemement coefficient,

$$\mathcal{E}(\mu_1) := \frac{2\pi^2}{1+(2\mu_1-1)^2}\left[1+\sqrt{1+(2\mu_1-1)^2}\right] - \frac{2}{\pi}[J(\mu_1, 1-\mu_1) + J(1-\mu_1, \mu_1)], \tag{1.12}$$

with

$$\begin{aligned} J(\mu_1,\mu_2) &:= \int d\mathbf{q}_2 \left[\int d\mathbf{q}_1\, \mathrm{Exp}[-\tfrac{1}{2}(\mu_1^2+\mu_2^2)(\mathbf{q}_1+\mathbf{q}_2)^2 - (\mu_2\mathbf{q}_1 - \mu_1\mathbf{q}_2)^2 - \mathbf{q}_1^2/2] \right. \\ &\left. I_0(|\mu_1-\mu_2|\, |\mathbf{q}_1+\mathbf{q}_2|\, |\mu_2\mathbf{q}_1 - \mu_1\mathbf{q}_2|)\right]^2, \end{aligned} \tag{1.13}$$

where I_0 is the modified Bessel function [2].

Observe that $\mathcal{E}(\mu_1) = \mathcal{E}(1-\mu_1)$, as it should be, because $\mathcal{P}(\varphi_{\text{out}})$ is invariant under the exchange of particles one and two.

Note that $J(1/2, 1/2) = \pi^3$. In the appendix we prove explicitly that $J(1,0) = 16.6377$. For $\mu_1 \in [0,1] \setminus \{1/2, 1\}$ we compute $J(\mu_1, 1-\mu_1)$ numerically using Gaussian quadratures.

The entanglement coefficient $\mathcal{E}(\mu_1)$ is universal in the sense that is independent of the interaction potential. Of course, this is only true if the potential is not identically zero, see the low-energy estimate for the scattering matrix given in (2.11). This is strikingly different from the three dimensional case that we previously studied in [3], where the entanglement created by the collision is proportional to the total scattering cross section.

Table 1 and Figure 1 show that -as in three dimensions [3]- the entanglement coefficient depends strongly in the difference of the masses. It takes its minimum for $\mu_1 = 0.5$, when the masses are equal, and it increases rapidly with the difference of the masses, as μ_1 tends to one. This implies that in experimental devices intended to create entanglement by collisions it is much more convenient to use particles with a large mass difference. There is a physically intuitive reason for this: in the collision of a heavy particle with a light one, the trajectory of the light particle is strongly changed, and there is large exchange of quantum information between the particles, what produces a large entanglement creation.

It is also interesting to remark that in the scattering of a particle with a large mass and a particle with a small mass we can assume that the trajectory of the large particle is not affected by the interaction, i.e. that, to a good approximation, it follows a free trajectory, and that the small particle feels a (external) interaction potential centered in the position of the large particle. However, the trajectory of the small particle will be strongly affected by the interaction, what will produce exchange of information between the particles, leading to the creation of entanglement between them. To evaluate this entanglement it is, however, necessary to take into account the degrees of freedom of both particles, as we do to compute the purity.

Besides its intrinsic interest, there are many reasons why it is important to study the creation of entanglement in scattering processes. From the conceptual point of view scattering is probably the simplest way to entangle two particles. Before the scattering, in the incoming state, the two particles are in a pure product state where they are uncorrelated. As they approach each other they become entangled by sharing quantum information between them. After the scattering, they are far apart from each other, but they remain entangled in the outgoing asymptotic state, that is not anymore a product state. Scattering is a basic dynamical process that is essential across all areas of physics. Furthermore, scattering interactions are fundamental at all scales and there is a large variety of scattering systems. As is well known, entanglement is a central notion of modern quantum theory. It is the fundamental resource for quantum information theory and quantum computation. It is a measure for quantum correlations between subsystems. In the case of bipartite systems in pure states, entanglement is a measure of how far away from being a product state a pure state of the bipartite system is. Currently it is well understood that entanglement in a pure bipartite quantum

state is equivalent to the degree of mixedness of each subsystem. For information about this issue see [5], [6], [7]. Moreover, the study of entanglement creation in scattering is interesting for a many other reasons. For example, for the implementation of quantum information processes in physical systems where scattering is central to the dynamics, like ultracold atoms and solid state devices. Moreover, the study of entanglement in the scattering of particles requires quantum information theory with continuous variables and mixed continuous-discrete variables. See [7] for a review of this topic. It is possible that scattering will provide a new perspective to quantum information theory. Finally, entanglement creation is important to the theory of scattering itself, because it poses new problems that can shed new light and new points of view in the study of scattering processes.

For previous results in the generation of entanglement in scattering processes in one dimension, mainly for potentials with explicit solution, see [4], [8], and the references quoted there. For the three dimensional case see [3]. Furthermore, [9], [10], [11], and the references quoted there, consider a system of heavy and light particles. They study the asymptotic dynamics and the decoherence produced on the heavy particles by the scattering with light particles in the limit of small mass ratio. This problem is different from the one that we discuss here. The loss of quantum coherence induced on heavy particles by the interaction with light ones has attracted much interest. See for example [12], and [13].

The paper is organized as follows. In Section 2 we study the low-energy asymptotics of the scattering matrix for the relative motion of the particles. In Section 3 we prove our results in the creation of entanglement. In Section 4 we give our conclusions. In the Appendix we explicitly evaluate integrals that need. Along the paper we denote by C a generic positive constant that does not necessarily have the same value in different appearances.

2 Scattering at Low-Energy in Two Dimensions

We denote by $\hat{\mathcal{H}} := L^2(\mathbb{R}^4)$ the state space in the momentum representation. The momentum of the particles one and two are, respectively, $\mathbf{p}_1, \mathbf{p}_2$. It convenient to take as coordinates in the momentum representation the momentum of the center of mass and the relative momentum,

$$\begin{array}{l} \mathbf{p}_{\text{cm}} := \mathbf{p}_1 + \mathbf{p}_2, \\ \mathbf{p} := \frac{m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2}{m_1 + m_2}. \end{array} \tag{2.1}$$

The state space in the momentum representation factorizes as a tensor product,

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{cm}} \otimes \hat{\mathcal{H}}_{\text{rel}}, \tag{2.2}$$

where $\hat{\mathcal{H}}_{\text{cm}} = L^2(\mathbb{R}^2), \hat{\mathcal{H}}_{\text{rel}} := L^2(\mathbb{R}^2)$ are, respectively, the state spaces in the momentum representation for the center-of-mass motion and the relative motion.

Since the potential depends on the difference of the coordinates of particles one and two the scattering matrix for the system decomposes as the tensor product $I_{cm} \otimes \mathcal{S}(\mathbf{p}^2/2m)$ of the identity on $\hat{\mathcal{H}}_{\text{cm}}$ times the scattering matrix for

the relative motion, $\mathcal{S}(\mathbf{p}^2/2m)$, in $\hat{\mathcal{H}}_{\rm rel}$, where m is the relative mass,

$$m := \frac{m_1 m_2}{m_1 + m_2}. \tag{2.3}$$

The scattering matrix $\mathcal{S}(\mathbf{p}^2/2m)$ is a unitary operator in $L^2(\mathbb{S}^1)$ for each $\mathbf{p}^2/2m \in (0, \infty)$, where we denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$.

We introduce some notation that we need. We denote $v := \sqrt{|V(x)|}$. Let P, Q be the projector operators in $L^2(\mathbb{R}^2)$,

$$P := \frac{1}{\alpha} v(\mathbf{x})\,(\cdot, v)\,, \quad Q := 1 - P, \tag{2.4}$$

where,

$$\alpha := \int_{\mathbb{R}^2} |V(\mathbf{x})|\, d\mathbf{x}. \tag{2.5}$$

Furthermore,

$$U(\mathbf{x}) := \begin{cases} 1, & \text{if } V(\mathbf{x}) \geq 0, \\ -1 & \text{if } V(\mathbf{x}) < 0. \end{cases} \tag{2.6}$$

By $\mathbf{M_{00}}$ we denote the integral operator with kernel,

$$M_{00}(\mathbf{x}, \mathbf{y}) := U(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{2\pi} \frac{2m}{\hbar^2}\, v(\mathbf{x})\, \ln \left(\frac{e^{\gamma}\, |\mathbf{x} - \mathbf{y}|}{2} \right)\, v(\mathbf{y}), \tag{2.7}$$

where γ is Euler's constant. Moreover, by $\mathbf{N_{00}}$ we denote the integral operator with kernel,

$$N_{0,0}(\mathbf{x}, \mathbf{y}) := U(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{2\pi} \frac{2m}{\hbar^2}\, v(\mathbf{x})\, \ln\left(|\mathbf{x} - \mathbf{y}| \right)\, v(\mathbf{y}), \tag{2.8}$$

and

$$D_0 := (Q\mathbf{M_{00}}Q)^{-1}, \text{ a bounded operator, } QL^2(\mathbb{R}^2) \to QL^2(\mathbb{R}^2). \tag{2.9}$$

The assumption that 0 is neither a resonance nor an eigenvalue for $H_{\rm rel}$ precisely means that $(Q\mathbf{M_{00}}Q)$ is invertible on $QL^2(\mathbb{R}^2)$ with bounded inverse.

Finally, we designate,

$$Y_0(\nu) := \frac{1}{\sqrt{2\pi}}, \nu \in \mathbb{S}^1. \tag{2.10}$$

For X, Y Banach spaces we denote by $\mathcal{B}\,(X; Y)$ the Banach space of all bounded linear operators from X, into Y. In the case $X = Y$ we use the notation $\mathcal{B}(X)$. By TrA we designate the trace of the operator A.

THEOREM 2.1. *Suposse that Assumption 1.1 is satisfied and that at zero $H_{\rm rel}$ has neither a resonance (half-bound state) nor an eigenvalue. Then, in the norm of $\mathcal{B}\left(L^2(\mathbb{S}^1)\right)$ we have for $|\mathbf{p}/\hbar| \to 0$ the expansion,*

$$\mathcal{S}(\mathbf{p}^2/2m) = I + i\pi\, \frac{1}{\ln |\mathbf{p}/\hbar|}\, \Sigma + \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2} \right)\, \frac{1}{|\ln(|\mathbf{p}|/\hbar)|^2}\, \Sigma + O\left(\frac{1}{|\ln(|\mathbf{p}|/\hbar)|^3} \right), \tag{2.11}$$

where I is the identity operator on $L^2(\mathbb{S}^1)$,

$$\Sigma := (\cdot, Y_0)\; Y_0, \tag{2.12}$$

and a is the scattering length defined by

$$\frac{1}{a} := \frac{2\pi}{\alpha} \operatorname{Tr} \left[P\mathbf{N_{00}}P - P\mathbf{M_{00}}QD_{00}\mathbf{M_{00}} + P\mathbf{M_{00}}Q \right]. \tag{2.13}$$

Proof: Let us denote by $\mathcal{S}_1(\lambda)$ the scattering matrix for the Hamiltonian $H_1 := -\Delta + \frac{2m}{\hbar^2}V(\mathbf{x})$. It follows from an elementary argument that,

$$\mathcal{S}(\mathbf{p}^2/2m) = \mathcal{S}_1((\mathbf{p}/\hbar)^2). \tag{2.14}$$

Furthermore [14],

$$\mathcal{S}_1(\lambda) = I - 2\pi i \frac{2m}{\hbar^2} \Gamma(\lambda)\, v\, \left(M(\lambda)\right)^{-1} \, v\, \Gamma^*(\lambda), \tag{2.15}$$

where $\Gamma(\lambda)$ is the trace operator,

$$\left(\Gamma(\lambda)\varphi\right)(\nu) := \frac{1}{\sqrt{2}} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\sqrt{\lambda}\,\nu\cdot\mathbf{x}}\, \varphi(\mathbf{x})\, d\mathbf{x}, \quad \nu \in \mathbb{S}^1, \tag{2.16}$$

and

$$M(\lambda) := \left(U + \frac{2m}{\hbar^2} v(-\Delta - \lambda - i0)^{-1}\, v \right) \tag{2.17}$$

has a bounded inverse in $L^2(\mathbb{R}^1)$.

Moreover, for all $f \in L^2(\mathbb{S}^1)$,

$$\left\| v(\mathbf{x}) \left(\Gamma^* - \frac{1}{\sqrt{2}} \Sigma \right) f \right\|_{L^2(\mathbb{R}^2)} \leq \sqrt{\lambda}\, \frac{1}{2\sqrt{\pi}}\, \| v(\mathbf{x})\, |\mathbf{x}| \|_{L^2(\mathbb{R}^2)}\, \|f\|_{L^2(\mathbb{S}^1)}. \tag{2.18}$$

Furthermore, by Schwarz's inequality,

$$\|v(\mathbf{x})\,|\mathbf{x}|\|^2_{L^2(\mathbb{R}^2)} = \left\|V(\mathbf{x})\,|\mathbf{x}|^2\right\|_{L^1(\mathbb{R}^2)} \leq \left\|(1+|x|)^{\beta} V(\mathbf{x})\right\|_{L^2(\mathbb{R}^2)} \left\|(1+|\mathbf{x}|)^{-\beta+2}\right\|_{L^2(\mathbb{R}^2)} \leq C,$$

and it follows that,

$$v(\mathbf{x})\,\Gamma^*(\lambda) = \frac{1}{\sqrt{2}}\, v(\mathbf{x})\,\Sigma + O(\sqrt{\lambda}), \quad \lambda \to 0, \tag{2.19}$$

in the operator norm in $\mathcal{B}(L^2(\mathbb{S}^1), L^2(\mathbb{R}^2))$. Taking the adjoint in both sides of (2.19) we obtain that,

$$\Gamma(\lambda)\, v(\mathbf{x}) = \frac{1}{\sqrt{2}} \frac{1}{2\pi}\, (\cdot, v) + O(\sqrt{\lambda}), \quad \lambda \to 0, \tag{2.20}$$

in the operator norm in $\mathcal{B}(L^2(\mathbb{R}^2), L^2(\mathbb{S}^1))$. By (2.15), (2.19), (2.20),

$$\mathcal{S}_1(\lambda) = I - \frac{i}{2} \frac{2m}{\hbar^2} \left(\left(M(\lambda)\right)^{-1} v,\, v \right) \Sigma + O(\sqrt{\lambda}), \quad \lambda \to 0, \tag{2.21}$$

in the norm of $\mathcal{B}\left(L^2(\mathbb{S}^1)\right)$. Equation (2.11) follows from (2.21) and Theorem 6.2 of [1].

□

The low-energy expansion (2.11) was previously proved by [15] in the case of exponentially decreasing potentials such that $\int V(\mathbf{x}) \neq 0$.

3 The Creation of Entanglement at Low-Energy

In what follows we consider a pure state of the two-particle system. The wave function in the momentum representation is given by $\varphi(\mathbf{p}_1, \mathbf{p}_2)$. We designate by $\rho(\varphi)$ the one-particle reduced density matrix with integral kernel,

$$\rho(\varphi)(\mathbf{p}_1, \mathbf{p}_1') := \int \varphi(\mathbf{p}_1, \mathbf{p}_2)\, \overline{\varphi(\mathbf{p}_1', \mathbf{p}_2)}\, d\mathbf{p}_2.$$

The purity, $\mathcal{P}(\varphi)$, is given by,

$$\mathcal{P}(\varphi) := \mathrm{Tr}(\rho^2) = \int d\mathbf{p}_1\, d\mathbf{p}_1'\, d\mathbf{p}_2\, d\mathbf{p}_2'\, \varphi(\mathbf{p}_1, \mathbf{p}_2)\, \overline{\varphi(\mathbf{p}_1', \mathbf{p}_2)}\, \varphi(\mathbf{p}_1', \mathbf{p}_2')\, \overline{\varphi(\mathbf{p}_1, \mathbf{p}_2')}. \tag{3.1}$$

As is well known [7, 16, 17], the purity is a measure of entanglement that is closely related to the Rényi entropy of order 2,$-\ln \mathrm{Tr}(\rho^2)$. Moreover, It is trivially related to the linear entropy, S_L, as $S_L = 1 - \mathcal{P}$. Clearly, It satisfies $0 \leq \mathcal{P} \leq 1$ if φ is normalized to one. Furthermore, it is equal to one for a product state, $\varphi = \varphi_1(\mathbf{p}_1)\, \varphi_2(\mathbf{p}_2)$. As we will show, the purity can be directly computed in terms of the scattering matrix. For this reason it is a measure of entanglement that is convenient for the study of entanglement creation in scattering processes.

We consider an incoming asymptotic state, in the center-of-mass frame, that is a product of two normalized Gaussian wave functions,

$$\varphi_{\mathrm{in}, \mathbf{p}_0}(\mathbf{p}_1, \mathbf{p}_2) := \varphi_{\mathbf{p}_0}(\mathbf{p}_1)\, \varphi_{-\mathbf{p}_0}(\mathbf{p}_2), \tag{3.2}$$

where

$$\varphi_{\mathbf{p}_0}(\mathbf{p}_1) := \frac{1}{(\sigma^2 \pi)^{1/2}} e^{-(\mathbf{p}_1 - \mathbf{p}_0)^2 / 2\sigma^2}. \tag{3.3}$$

Note that in our incoming asymptotic state (3.2) particle one has mean momentum $\mathbf{p}_0$ and particle two has mean momentum $-\mathbf{p}_0$. The variance of the momentum distribution of both particles is σ. We assume that the scattering takes place at the origin at time zero, and for this reason the average position of both particles is zero in the incoming asymptotic state (3.2). Observe that by (2.1) the mean value of the relative momentum in the state (3.2) is equal to $\mathbf{p}_0$.

Since the incoming asymptotic state $\varphi_{\mathrm{in}, \mathbf{p}_0}$ is a product state its purity is one,

$$\mathcal{P}(\varphi_{\mathrm{in}, \mathbf{p}_0}) = 1. \tag{3.4}$$

The outgoing asymptotic state of the two particles, $\varphi_{\mathrm{out}, \mathbf{p}_0}$ -after the scattering process is over- is given by

$$\varphi_{\mathrm{out}, \mathbf{p}_0}(\mathbf{p}_1, \mathbf{p}_2) := \left(\mathcal{S}(\mathbf{p}^2/2m) \varphi_{\mathrm{in}, \mathbf{p}_0} \right)(\mathbf{p}_1, \mathbf{p}_2). \tag{3.5}$$

As the relative momentum $\mathbf{p}$ depends on $\mathbf{p}_1$ and on $\mathbf{p}_2$, $\varphi_{\mathrm{out}, \mathbf{p}_0}$ is not a product state, and then it has purity smaller than one. This implies that the scattering process has created entanglement between the two particles.

Below we give a rigorous computation of the leading order of the purity of the outgoing asymptotic state $\varphi_{\mathrm{out}, \mathbf{p}_0}$ in the low-energy limit for the relative motion. To be in the low-energy regime the following two conditions have to

be satisfied. 1. The mean relative momentum $\mathbf{p}_0$ has to be small. 2. The variance σ has to be small. Note that if σ is large the incoming asymptotic state $\varphi_{\mathrm{in},\mathbf{p}_0}$ has a big probability of having large momentum, even if the mean relative momentum $\mathbf{p}_0$ is small.

Let us introduce some notations that we use later.

We denote by φ_{in} the incoming asymptotic state with mean value of the relative momentum zero,

$$\varphi_{\mathrm{in}}(\mathbf{p}_1, \mathbf{p}_2) := \varphi(\mathbf{p}_1)\, \varphi(\mathbf{p}_2), \tag{3.6}$$

where,

$$\varphi(\mathbf{p}) := \frac{1}{(\sigma^2 \pi)^{1/2}} e^{-\mathbf{p}^2/2\sigma^2}, \tag{3.7}$$

and by φ_{out} the outgoing asymptotic state with incoming asymptotic state φ_{in},

$$\varphi_{\mathrm{out}}(\mathbf{p}_1, \mathbf{p}_2) := \left(\mathcal{S}(\mathbf{p}^2/2m)\varphi_{\mathrm{in}}\right)(\mathbf{p}_1, \mathbf{p}_2). \tag{3.8}$$

Recall that,

$$\mu_i = \frac{m_i}{m_1 + m_2}, i = 1, 2, \tag{3.9}$$

is the ratio of the mass of the i particle to the total mass.

It follows from (2.1) that,

$$\mathbf{p}_1 = \mu_1 \mathbf{p}_{\mathrm{cm}} + \mathbf{p}, \tag{3.10}$$

$$\mathbf{p}_2 = \mu_2 \mathbf{p}_{\mathrm{cm}} - \mathbf{p}. \tag{3.11}$$

We prepare some results that we will use. It follows from (3.2, 3.3, 3.10, 3.11) that

$$\varphi_{\mathrm{in},\mathbf{p}_0} = \frac{1}{\sigma^2 \pi}\, e^{-(\mu_1^2+\mu_2^2)\mathbf{p}_{\mathrm{cm}}^2/2\sigma^2}\, e^{-(\mathbf{p}-\mathbf{p}_0)^2/\sigma^2}\, e^{-(\mu_1-\mu_2)\mathbf{p}_{\mathrm{cm}}\cdot(\mathbf{p}-\mathbf{p}_0)}. \tag{3.12}$$

REMARK 3.1. *For some positive constant δ,*

$$|\varphi_{\mathrm{in}}| \leq \frac{1}{\sigma^2 \pi}\, e^{-\delta(\mathbf{p}_{\mathrm{cm}}^2 + \mathbf{p}^2)/2\sigma^2}. \tag{3.13}$$

Proof: Note that $\mu_1^2 + \mu_2^2 = \frac{1}{2}\left(1 + (\mu_1 - \mu_2)^2\right)$. Then, for $\alpha \geq 0$,

$$\begin{aligned} &(\mu_1^2 + \mu_2^2)\mathbf{p}_{\mathrm{cm}}^2/2\sigma^2 + \mathbf{p}^2/\sigma^2 + (\mu_1 - \mu_2)\mathbf{p}_{\mathrm{cm}} \cdot \mathbf{p} \geq \\ &\left(\tfrac{1}{4} + (\mu_1 - \mu_2)^2(\tfrac{1}{4} - \tfrac{\alpha}{2})\right) \tfrac{\mathbf{p}_{\mathrm{cm}}^2}{\sigma^2} + \tfrac{\mathbf{p}^2}{\sigma^2}\left(1 - \tfrac{1}{2\alpha}\right) \geq \delta(\mathbf{p}_{\mathrm{cm}}^2 + \mathbf{p}^2)/2\sigma^2, \end{aligned} \tag{3.14}$$

provided that we choose α so that, $0 < \delta/2 \leq \min[\frac{1}{4} + (\mu_1 - \mu_2)^2(\frac{1}{4} - \frac{\alpha}{2}), (1 - \frac{1}{2\alpha})]$. The remark follows from (3.12) and (3.14).

PROPOSITION 3.2. *For any $\alpha, \beta \geq 0$,*

$$\left\| \frac{(\ln(2 + |\mathbf{p}|/\hbar))^\alpha}{(1 + |\ln(|\mathbf{p}|/\hbar)|)^\beta}\ \varphi_{\mathrm{in},\mathbf{p}_0} \right\| = O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^\beta}\right), \qquad \text{as}\, \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0, \tag{3.15}$$

$$\left\| \frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{|\ln(|\mathbf{p}|/\hbar)|^\beta} \; \varphi_{\rm in} \right\| = O\left(\frac{1}{|\ln(\sigma/\hbar)|^\beta} \right), \qquad \text{as } \sigma/\hbar \to 0. \tag{3.16}$$

Furthermore, uniformly for $|\mathbf{p}_0|/\sigma$ *in bounded sets,*

$$\left\| \frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta} \; (\varphi_{{\rm in},\mathbf{p}_0} - \varphi_{\rm in}) \right\| \leq \frac{|\mathbf{p}_0|}{\sigma} \, O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^\beta} \right), \qquad \text{as} \, \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0. \tag{3.17}$$

Proof: By (3.13) we have that,

$$\left\| \frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta} \; \varphi_{{\rm in},\mathbf{p}_0} \right\|^2 \leq I_1 + I_2, \tag{3.18}$$

where, for some $1 > \gamma > 0$,

$$I_1 := \int_{|\mathbf{p}|/\sigma \geq 1/(\sigma/\hbar + |\mathbf{p}_0|/\hbar)^\gamma} \left(\frac{(\ln(2+|\mathbf{p}+\mathbf{p}_0|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}+\mathbf{p}_0|/\hbar)|)^\beta} \right)^2 \frac{1}{\sigma^4 \pi^2} \, e^{-\delta(\mathbf{p}_{\rm cm}^2 + \mathbf{p}^2)/\sigma^2} \, d\mathbf{p}_{\rm cm} \; d\mathbf{p} \leq C_N (\sigma/\hbar + |\mathbf{p}_0|/\hbar)^N, \tag{3.19}$$

$N = 1, 2, \cdots$, and

$$I_2 := \int_{|\mathbf{p}|/\sigma \leq 1/(\sigma/\hbar + |\mathbf{p}_0|/\hbar)^\gamma} \left(\frac{(\ln(2+|\mathbf{p}+\mathbf{p}_0|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}+\mathbf{p}_0|/\hbar)|)^\beta} \right)^2 \frac{1}{\sigma^4 \pi^2} \, e^{-\delta(\mathbf{p}_{\rm cm}^2 + \mathbf{p}^2)/\sigma^2} \, d\mathbf{p}_{\rm cm} \; d\mathbf{p}. \tag{3.20}$$

Moreover, for $|\mathbf{p}|/\sigma \leq 1/(\sigma/\hbar + |\mathbf{p}_0|/\hbar)^\gamma$ and $(\sigma/\hbar + |\mathbf{p}_0|/\hbar)^{1-\gamma} \leq 1/2$,

$$\frac{1}{(1+|\ln(|\mathbf{p}+\mathbf{p}_0/\hbar|)|} \leq \frac{1}{(1+(1-\gamma)|\ln((\sigma/\hbar + |\mathbf{p}_0|/\sigma) \; 2^{1/(1-\gamma)}|},$$

and then,

$$I_2 = O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^\beta} \right), \qquad \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0. \tag{3.21}$$

Equation (3.15) follows from (3.19) and (3.21).

In the same way, it follows from (3.13) that,

$$\left\| \frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{|\ln(|\mathbf{p}|/\hbar)|^\beta} \; \varphi_{\rm in} \right\|^2 \leq I_1 + I_2, \tag{3.22}$$

where, for some $1 > \gamma > 0$,

$$I_1 := \int_{|\mathbf{p}|/\sigma \geq 1/(\sigma/\hbar)^\gamma} \left(\frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{|\ln(|\mathbf{p}|/\hbar)|^\beta} \right)^2 \frac{1}{\sigma^4 \pi^2} \, e^{-\delta(\mathbf{p}_{\rm cm}^2 + \mathbf{p}^2)/\sigma^2} \, d\mathbf{p}_{\rm cm} \; d\mathbf{p} \leq C_N (\sigma/\hbar)^N, \qquad N = 1, 2, \cdots, \tag{3.23}$$

and

$$I_2 := \int_{|\mathbf{p}|/\sigma \leq 1/(\sigma/\hbar)^\gamma} \left(\frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{|\ln(|\mathbf{p}|/\hbar)|^\beta} \right)^2 \frac{1}{\sigma^4 \pi^2} \, e^{-\delta(\mathbf{p}_{\rm cm}^2 + \mathbf{p}^2)/\sigma^2} \, d\mathbf{p}_{\rm cm} \; d\mathbf{p}. \tag{3.24}$$

As above, for $|\mathbf{p}|/\sigma \leq 1/(\sigma/\hbar)^\gamma$ and $(\sigma/\hbar)^{1-\gamma} \leq 1$,

$$\frac{1}{|\ln|\mathbf{p}/\hbar||} \leq \frac{1}{(1-\gamma)|\ln(\sigma/\hbar)|},$$

and then,

$$I_2 = O\left(\frac{1}{|\ln(\sigma/\hbar)|^\beta}\right), \qquad \sigma/\hbar \to 0. \tag{3.25}$$

Equation (3.16) follows from (3.23) and (3.25).

We now prove (3.17). We first consider the case when $|\mathbf{p}_0|/\sigma \leq 1$. By (3.12, 3.13),

$$\begin{aligned}
&|\varphi_{\text{in},\mathbf{p}_0} - \varphi_{\text{in}}| \leq \tfrac{1}{\sigma^2\pi}\, e^{-\delta(\mathbf{p}_{\text{cm}}^2+\mathbf{p}^2)/2\sigma^2} \left| e^{-(\mathbf{p}_0^2+2\mathbf{p}\cdot\mathbf{p}_0+(\mu_1-\mu_2)\mathbf{p}_{\text{cm}}\cdot\mathbf{p}_0)/\sigma^2} - 1 \right| \\
&\leq \tfrac{1}{\sigma^2\pi}\, e^{-\delta(\mathbf{p}_{\text{cm}}^2+\mathbf{p}^2)/2\sigma^2} e^{(\mathbf{p}_0^2+2|\mathbf{p}||\mathbf{p}_0|+|(\mu_1-\mu_2)||\mathbf{p}_{\text{cm}}||\mathbf{p}_0|)/\sigma^2}\, (\mathbf{p}_0^2+2|\mathbf{p}||\mathbf{p}_0|+|(\mu_1-\mu_2)||\mathbf{p}_{\text{cm}}||\mathbf{p}_0|)/\sigma^2 \leq \\
&\tfrac{1}{\sigma^2\pi}\, e^{-\delta(\mathbf{p}_{\text{cm}}^2+\mathbf{p}^2)/2\sigma^2} e^{(1+2|\mathbf{p}|/\sigma+|\mathbf{p}_{\text{cm}}|/\sigma)}\, |\mathbf{p}_0/\sigma|\,(1+2|\mathbf{p}|/\sigma+|\mathbf{p}_{\text{cm}}|/\sigma)\,.
\end{aligned} \tag{3.26}$$

Then,

$$\begin{aligned}
&\left\| \tfrac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta}\ (\varphi_{\text{in},\mathbf{p}_0} - \varphi_{\text{in}}) \right\|^2 \leq \tfrac{1}{\sigma^4\pi^2} \int \tfrac{(\ln(2+|\mathbf{p}|/\hbar))^{2\alpha}}{(1+|\ln(|\mathbf{p}|/\hbar)|)^{2\beta}}\, e^{-\delta(\mathbf{p}_{\text{cm}}^2+\mathbf{p}^2)/\sigma^2} \\
&e^{2(1+2|\mathbf{p}|/\sigma+|\mathbf{p}_{\text{cm}}|/\sigma)}\, |\mathbf{p}_0/\sigma|^2\, (1+2|\mathbf{p}|/\sigma+|\mathbf{p}_{\text{cm}}|/\sigma)^2\, d\mathbf{p}_{\text{cm}}\, d\mathbf{p}.
\end{aligned}$$

Estimating as in equations (3.18- 3.21) with $\mathbf{p}_0 = 0$, we prove that for $|\mathbf{p}_0|/\sigma \leq 1$,

$$\left\| \frac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta}\ (\varphi_{\text{in},\mathbf{p}_0} - \varphi_{\text{in}}) \right\| \leq |\mathbf{p}_0/\sigma|\ O\left(\frac{1}{|\ln(\sigma/\hbar)|^\beta}\right) \leq |\mathbf{p}_0/\sigma|\ O\left(\frac{1}{|\ln(\sigma/\hbar+|\mathbf{p}_0|/\hbar)|^\beta}\right), \text{as } \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0.$$

In the case $|\mathbf{p}_0/\sigma| \geq 1$ the estimate is immediate from (3.15), because,

$$\begin{aligned}
&\left\| \tfrac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta}\ (\varphi_{\text{in},\mathbf{p}_0} - \varphi_{\text{in}}) \right\| \leq \left\| \tfrac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta}\ \varphi_{\text{in},\mathbf{p}_0} \right\| + \left\| \tfrac{(\ln(2+|\mathbf{p}|/\hbar))^\alpha}{(1+|\ln(|\mathbf{p}|/\hbar)|)^\beta}\ \varphi_{\text{in}} \right\| = O\left(\tfrac{1}{|\ln(\sigma/\hbar+|\mathbf{p}_0|/\hbar)|^\beta}\right) \\
&\leq |\mathbf{p}_0/\sigma|\ \ O\left(\tfrac{1}{|\ln(\sigma/\hbar+|\mathbf{p}_0|/\hbar)|^\beta}\right), \qquad \text{as } \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0.
\end{aligned}$$

□

We define,

$$\mathcal{T}(\mathbf{p}^2/2m) := \mathcal{S}(\mathbf{p}^2/m) - I + i\pi\, \frac{1}{1+|\ln(|\mathbf{p}|/\hbar)|}, \tag{3.27}$$

where I is the identity operator on $L^2(\mathbb{S}^1)$. It follows from (2.11) and since $\|\mathcal{S}(\mathbf{p}^2/2m)\|_{\mathcal{B}(L^2(\mathbb{S}^1))} = 1$, that

$$\left\|\mathcal{T}(\mathbf{p}^2/2m)\right\|_{\mathcal{B}(L^2(\mathbb{S}^1))} \leq C\frac{(\ln(2+|\mathbf{p}|/\hbar))^2}{(1+|\ln(|\mathbf{p}|/\hbar)|)^2}. \tag{3.28}$$

Hence by (3.15),

$$\left\|\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\text{in},\mathbf{p}_0}\right\| = O\left(\frac{1}{|\ln(\sigma/\hbar+|\mathbf{p}_0|/\hbar)|^2}\right), \qquad \text{as}\, \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0. \tag{3.29}$$

Let us denote,

$$\mathcal{L}(\phi_1,\phi_2,\phi_3,\phi_4) := \int\ d\mathbf{p}_1\, d\mathbf{p}_1'\, d\mathbf{p}_2\, d\mathbf{p}_2'\, \phi_1(\mathbf{p}_1,\mathbf{p}_2)\, \overline{\phi_2(\mathbf{p}_1',\mathbf{p}_2)}\, \phi_3(\mathbf{p}_1',\mathbf{p}_2')\, \overline{\phi_4(\mathbf{p}_1,\mathbf{p}_2')}. \tag{3.30}$$

We have that,

$$\mathcal{P}(\phi) = \mathcal{L}(\phi,\phi,\phi,\phi).$$

The Schwarz inequality implies that,

$$|\mathcal{L}(\phi_1, \phi_2, \phi_3, \phi_4)| \leq \Pi_{j=1}^4 \|\phi_j\|. \tag{3.31}$$

We state below our first result in the low-energy behaviour of the purity.

THEOREM 3.3. *Suposse that Assumption 1.1 is satisfied and that at zero $H_{\rm rel}$ has neither a resonance (half-bound state) nor an eigenvalue. Then, uniformly for $|\mathbf{p}_0|/\sigma$in bounded sets,*

$$\mathcal{P}(\varphi_{\text{out},\mathbf{p}_0}) = \mathcal{P}(\varphi_{\text{out}}) + \frac{|\mathbf{p}_0|}{\sigma}\, O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right), \qquad \text{as}\, \sigma/\hbar + |\mathbf{p}_0|/\hbar \to 0. \tag{3.32}$$

Proof: Writing $\varphi_{\text{out},\mathbf{p}_0}$ as,

$$\varphi_{\text{out},\mathbf{p}_0} := \mathcal{S}(\mathbf{p}^2/2m)\varphi_{\text{in},\mathbf{p}_0} = \varphi_{\text{in},\mathbf{p}_0} - i\pi\, \frac{1}{1 + |\ln(|\mathbf{p}|/\hbar)|} \varphi_{\text{in},\mathbf{p}_0} + \mathcal{T}(\mathbf{p}^2/2m)\varphi_{\text{in},\mathbf{p}_0},$$

and using (3.4), we see that we can write $\mathcal{P}(\varphi_{\text{out},\mathbf{p}_0})$ as follows,

$$\mathcal{P}(\varphi_{\text{out},\mathbf{p}_0}) = 1 + \sum_{i=1}^{4} \mathcal{L}_{1,i}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) + \mathcal{R}(\mathbf{p}_0), \tag{3.33}$$

where

$$\mathcal{L}_{1,i}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\psi_1, \psi_2, \psi_3, \psi_4), \tag{3.34}$$

where one of the ψ_j is equal to $\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\text{in},\mathbf{p}_0}$ and the remaining 3 are equal to $\varphi_{\text{in},\mathbf{p}_0}$. Similarly,

$$\mathcal{R}(\mathbf{p}_0) := \sum_{i=1}^{A} \mathcal{L}_{2,i}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4), \tag{3.35}$$

for some integer A, and where each of the $\mathcal{L}_{2,i}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4)$ is equal to,

$$\mathcal{L}_{2,i}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\psi_1, \psi_2, \psi_3, \psi_4), \tag{3.36}$$

where for some $2 \leq k \leq 4$, k of the ψ_j are equal either to $-i\pi\, \frac{1}{1+|\ln|\mathbf{p}/\hbar||} \varphi_{\text{in},\mathbf{p}_0}$ or to $\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\text{in},\mathbf{p}_0}$ and the remaining $4-k$ are equal to $\varphi_{\text{in},\mathbf{p}_0}$. Similarly,

$$\mathcal{P}(\varphi_{\text{out}}) = 1 + \sum_{i=1}^{4} \mathcal{L}_{1,i}(0, \psi_1, \psi_2, \psi_3, \psi_4) + \mathcal{R}(0), \tag{3.37}$$

with

$$\mathcal{R}(0) := \sum_{i=1}^{A} \mathcal{L}_i(0, \psi_1, \psi_2, \psi_3, \psi_4). \tag{3.38}$$

Below we prove that,

$$\mathcal{L}_{1,i}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}_{1,i}(0, \psi_1, \psi_2, \psi_3, \psi_4) + \frac{|\mathbf{p}_0|}{\sigma}\, O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right), \quad i = 1, 2, 3, 4, \tag{3.39}$$

$$\mathcal{R}(\mathbf{p}_0) = \mathcal{R}(0) + \frac{|\mathbf{p}_0|}{\sigma}\, O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right), \tag{3.40}$$

what proves the theorem in view of (3.33,3.37).

We proceed to prove (3.39). Without losing generality we can assume that,

$$\mathcal{L}_{1,1}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}). \tag{3.41}$$

We have that,

$$\begin{aligned}&\mathcal{L}_{1,1}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\mathrm{in}}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0})+\\ &\mathcal{L}(\mathcal{T}(\mathbf{p}^2/2m)(\varphi_{\mathrm{in},\mathbf{p}_0} - \varphi_{\mathrm{in}}), \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}).\end{aligned} \tag{3.42}$$

By (3.17, 3.28, 3.31, 3.42),

$$\mathcal{L}_{1,1}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\mathrm{in}}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}) + \frac{|\mathbf{p}_0|}{\sigma} O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right). \tag{3.43}$$

In the same way, using (3.17, 3.29, 3.43), we prove that,

$$\mathcal{L}_{1,1}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\mathcal{T}(\mathbf{p}^2/2m)\varphi_{\mathrm{in}}, \varphi_{\mathrm{in}}, \varphi_{\mathrm{in},\mathbf{p}_0}, \varphi_{\mathrm{in},\mathbf{p}_0}) + \frac{|\mathbf{p}_0|}{\sigma} O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right). \tag{3.44}$$

Repeating this argument two more times we obtain that,

$$\mathcal{L}_{1,1}(\mathbf{p}_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}_{1,1}(0, \psi_1, \psi_2, \psi_3, \psi_4) + \frac{|\mathbf{p}_0|}{\sigma} O\left(\frac{1}{|\ln(\sigma/\hbar + |\mathbf{p}_0|/\hbar)|^2}\right). \tag{3.45}$$

We prove in the same way that (3.39) holds for $i = 2, 3, 4$. Furthermore, (3.40) is proven by the same argument.

□

The next theorem gives us the leading order of the purity of φ_{out} at low-energy.

THEOREM 3.4. *Suposse that Assumption 1.1 is satisfied and that at zero H_{rel} has neither a resonance (half-bound state) nor an eigenvalue. Then, as $\sigma/\hbar \to 0$.*

$$\mathcal{P}(\varphi_{\mathrm{out}}) = \mathcal{P}\left(\left[I + i\pi \frac{1}{\ln|\mathbf{p}/\hbar|}\Sigma + \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2}\right)\frac{1}{(\ln|\mathbf{p}/\hbar|)^2}\Sigma\right]\varphi_{\mathrm{in}}\right) + O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right). \tag{3.46}$$

Proof: We write φ_{out} as follows,

$$\varphi_{\mathrm{out}} = \varphi_{\mathrm{out},1} + \mathcal{T}_1(\mathbf{p}^2/2m)\varphi_{\mathrm{in}},$$

where,

$$\varphi_{\mathrm{out},1} := \left[I + i\pi\frac{1}{\ln|\mathbf{p}/\hbar|}\Sigma + \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2}\right)\frac{1}{(\ln|\mathbf{p}/\hbar|)^2}\Sigma\right]\varphi_{\mathrm{in}}, \tag{3.47}$$

and

$$\mathcal{T}_1 := S(\mathbf{p}^2/2m) - I - i\pi\frac{1}{\ln|\mathbf{p}/\hbar|}\Sigma - \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2}\right)\frac{1}{(\ln|\mathbf{p}/\hbar|)^2}\Sigma.$$

By (2.11) and since $\|\mathcal{S}(\mathbf{p}^2/2m)\|_{\mathcal{B}(L^2(\mathbb{S}^1))} = 1$,

$$\left\|\mathcal{T}_1(\mathbf{p}^2/2m)\right\|_{\mathcal{B}(L^2(\mathbb{S}^1))} \leq C\frac{(\ln(2 + |\mathbf{p}|/\hbar))^3}{|\ln(|\mathbf{p}|/\hbar)|^3}. \tag{3.48}$$

Using this decomposition we write $\mathcal{P}(\varphi_{\text{out}})$ as follows,

$$\mathcal{P}(\varphi_{\text{out}}) = \mathcal{P}(\varphi_{\text{out},1}) + \mathcal{R}_1(\sigma), \tag{3.49}$$

where $\mathcal{R}_1(\sigma)$ is given by,

$$\mathcal{R}_1(\sigma) := \sum_{i=1}^{D} \mathcal{L}_i(\sigma, \psi_1, \psi_2, \psi_3, \psi_4), \tag{3.50}$$

for some integer D, and where each of the $\mathcal{L}_i(\sigma, \psi_1, \psi_2, \psi_3, \psi_4)$ is equal to,

$$\mathcal{L}_i(\sigma, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\psi_1, \psi_2, \psi_3, \psi_4), \tag{3.51}$$

where for some $1 \leq k \leq 4$, k of the ψ_j are equal to $\varphi_{\text{out},1}$ and the remaining $4-k$ are equal to $\mathcal{T}_1(\mathbf{p}^2/2m)\varphi_{\text{in}}$.

We complete the proof of the theorem proving that,

$$\mathcal{R}_1(\sigma) = O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right), \qquad \text{as } \sigma/\hbar \to 0. \tag{3.52}$$

We can assume that,

$$\mathcal{L}_1(\sigma, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\varphi_{\text{out},1}, \varphi_{\text{out},1}, \varphi_{\text{out},1}, \mathcal{T}_1(\mathbf{p}^2/2m)\varphi_{\text{in}}). \tag{3.53}$$

By (3.16, 3.48) We have that,

$$\mathcal{L}_1(\sigma, \psi_1, \psi_2, \psi_3, \psi_4) = O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right), \qquad \text{as}\, \sigma/\hbar \to 0. \tag{3.54}$$

We estimate the remaining terms in (3.52) in the same way.

□

Let us denote,

$$\psi(\mathbf{q}) := \frac{1}{(\pi)^{1/2}} e^{-\mathbf{q}^2/2}, \mathbf{q} \in \mathbb{R}^2,$$

$$\psi_{\text{in}}(\mathbf{q}_1, \mathbf{q}_2) := \psi(\mathbf{q}_1)\, \psi(\mathbf{q}_2).$$

PROPOSITION 3.5. *For any $\alpha, \beta \geq 0$,*

$$\left\| \frac{(\ln|\mathbf{q}|)^\alpha}{|\ln(\sigma|\mathbf{q}|/\hbar)|^\beta}\; \psi_{\text{in}} \right\| = O\left(\frac{1}{|\ln(\sigma/\hbar)|^\beta}\right), \qquad \text{as } \sigma/\hbar \to 0. \tag{3.55}$$

Proof: We follow the proof of (3.16). By (3.13) with $\sigma = 1$,

$$\left\| \frac{(\ln|\mathbf{q}|)^\alpha}{|\ln(\sigma|\mathbf{q}|/\hbar)|^\beta}\; \psi_{\text{in}} \right\|^2 \leq I_1 + I_2, \tag{3.56}$$

where, for some $1 > \gamma > 0$,

$$I_1 := \int_{|\mathbf{q}|\geq 1/(\sigma/\hbar)^\gamma} \left| \frac{|\ln|\mathbf{q}||^{2\alpha}}{|\ln(\sigma|\mathbf{q}|/\hbar)|^{2\beta}} \right) \frac{1}{\pi^2} e^{-\delta(\mathbf{q}_{\rm cm}^2+\mathbf{q}^2)/} d\mathbf{q}_{\rm cm}\, d\mathbf{q} \leq C_N (\sigma/\hbar)^N, \qquad N=1,2,\cdots, \tag{3.57}$$

and

$$I_2 := \int_{|\mathbf{p}|\leq 1/(\sigma/\hbar)^\gamma} \left(\frac{(\ln|\mathbf{q}|)^{\alpha}}{|\ln(\sigma|\mathbf{q}|/\hbar)|^{\beta}} \right)^2 \frac{1}{\pi^2} e^{-\delta(\mathbf{q}_{\rm cm}^2+\mathbf{q}^2)} d\mathbf{q}_{\rm cm}\ d\mathbf{q}. \tag{3.58}$$

Furthermore, for $|\mathbf{q}| \leq 1/(\sigma/\hbar)^\gamma$ and $(\sigma/\hbar)^{1-\gamma} \leq 1$,

$$\frac{1}{|\ln(\sigma|\mathbf{q}|/\hbar)|} \leq \frac{1}{(1-\gamma)|\ln(\sigma/\hbar)|},$$

and then,

$$I_2 = O\left(\frac{1}{|\ln(\sigma/\hbar)|^\beta} \right), \qquad \sigma/\hbar \to 0. \tag{3.59}$$

Equation (3.55) follows from (3.57) and (3.59).

□

A straightforward computation with the help of (3.55) shows that,

$$\mathcal{P}\left(\left[I + i\pi \tfrac{1}{\ln|\mathbf{p}/\hbar|} \Sigma + \left(i\pi(\ln 2 - \gamma + \tfrac{1}{a}) - \tfrac{\pi^2}{2} \right) \tfrac{1}{(\ln|\mathbf{p}/\hbar|)^2} \Sigma \right] \varphi_{\rm in} \right) = 1 - \tfrac{1}{(\ln(\sigma/\hbar))^2} \left(\mathcal{P}_1(\psi_{\rm in}) + \mathcal{P}_2(\psi_{\rm in}) \right) + \\ O\left(\tfrac{1}{(\ln(\sigma/\hbar))^3} \right), \quad \text{as}\, \sigma/\hbar \to 0, \tag{3.60}$$

where,

$$\mathcal{P}_1(\psi_{\rm in}) = \Sigma_{j=1}^3 \mathcal{P}_{1,j}(\psi_{\rm in}), \tag{3.61}$$

with

$$\mathcal{P}_{1,1}(\psi_{\rm in}) = -2\pi^2 \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3\, (\Sigma\psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_2))\, (\Sigma\psi_{\rm in}(\mathbf{q}_3,\mathbf{q}_2))\, \psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_3), \tag{3.62}$$

$$\mathcal{P}_{1,2}(\psi_{\rm in}) = -2\pi^2 \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3\ (\Sigma\psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_2))\ (\Sigma\psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_3))\, \psi_{\rm in}(\mathbf{q}_2,\mathbf{q}_3), \tag{3.63}$$

$$\mathcal{P}_{1,3}(\psi_{\rm in}) = 2\pi^2 \left[\int d\mathbf{q}_1 d\mathbf{q}_2\ (\Sigma\psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_2))\ \psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_2) \right]^2, \tag{3.64}$$

and

$$\mathcal{P}_2(\psi_{\rm in}) = 2\pi^2 \int d\mathbf{q}_1 d\mathbf{q}_2\ (\Sigma\psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_2))\ \psi_{\rm in}(\mathbf{q}_1,\mathbf{q}_2). \tag{3.65}$$

Explicitly evaluating the integrals in (3.62, 3.63, 3.64) using (3.12), we prove that,

$$\mathcal{P}_{1,1}(\psi_{\rm in}) = -\frac{2}{\pi} J(\mu_1,\mu_2), \tag{3.66}$$

$$\mathcal{P}_{1,2}(\psi_{\rm in}) = -\frac{2}{\pi} J(\mu_2,\mu_1), \tag{3.67}$$

$$\mathcal{P}_{1,3}(\psi_{\rm in}) = 2\pi^2\ \left(L(\mu_1,\mu_2)\right)^2, \tag{3.68}$$

$$\mathcal{P}_2(\psi_{\rm in}) = 2\pi^2 L(\mu_1,\mu_2), \tag{3.69}$$

where,

$$J(\mu_1,\mu_2) := \int d\mathbf{q}_2 \left[\int d\mathbf{q}_1 \, \mathrm{Exp}[-\tfrac{1}{2}(\mu_1^2+\mu_2^2)(\mathbf{q}_1+\mathbf{q}_2)^2 - (\mu_2\mathbf{q}_1-\mu_1\mathbf{q}_2)^2 - \mathbf{q}_1^2/2] \right. \\ \left. I_0(|\mu_1-\mu_2|\,|\mathbf{q}_1+\mathbf{q}_2|\,|\mu_2\mathbf{q}_1-\mu_1\mathbf{q}_2|)\right]^2. \tag{3.70}$$

Here I_0 is the modified Bessel function [2], and

$$L(\mu_1,\mu_2) := \int_0^\infty \int_0^\infty d\lambda\, d\rho\, e^{-2\lambda}\, e^{-(\mu_1^2+\mu_2^2)\rho} \left(I_0(|\mu_1-\mu_2|\sqrt{\lambda\,\rho}\,)\right)^2. \tag{3.71}$$

We prove in the appendix that,

$$L(\mu_1,1-\mu_1) = \frac{1}{\sqrt{1+(2\mu_1-1)^2}}. \tag{3.72}$$

We denote by $\mathcal{E}(\mu_1)$ the entanglement coefficient,

$$\mathcal{E}(\mu_1) := 2\pi^2 L(\mu_1,1-mu_1)\,(1+L(\mu_1,1-mu_1)) - \tfrac{2}{\pi}[J(\mu_1,1-\mu_1)+J(1-\mu_1,\mu_1)] = \\ \frac{2\pi^2}{1+(2\mu_1-1)^2}\left[1+\sqrt{1+(2\mu_1-1)^2}\right] - \frac{2}{\pi}[J(\mu_1,1-\mu_1)+J(1-\mu_1,\mu_1)]. \tag{3.73}$$

The next theorem is our main result.

THEOREM 3.6. *Suppose that Assumption 1.1 is satisfied and that at zero $H_{\rm rel}$ has neither a resonance (half-bound state) nor an eigenvalue. Then,*

$$\mathcal{P}(\varphi_{\rm out}) = 1 - \frac{1}{(\ln(\sigma/\hbar))^2}\mathcal{E}(\mu_1) + O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right), \qquad \text{as}\, \sigma/\hbar \to 0, \tag{3.74}$$

where the entanglement coefficient $\mathcal{E}(\mu_1)$ is given by (3.73).

Proof: The theorem follows from (3.46, 3.60, 3.61, 3.66-3.69, 3.72).

□

Note that $\mathcal{E}(\mu_1) = \mathcal{E}(1-\mu_1)$, as it should be, because $\mathcal{P}(\varphi_{\rm out})$ is invariant under the exchange of particles one and two.

Observe that,

$$J(1/2,1/2) = \pi^3.$$

By (3.73) for $\mu_1 = 1/2$, when the masses are equal, the entanglement coefficient is zero, $\mathcal{E}(1/2) = 0$. Of course, this only means that in this case the purity is one at leading order.

We explicitly evaluate in the appendix $J(1,0)$,

$$J(1,0) = 16.6377. \tag{3.75}$$

For $\mu_1 \in [0,1] \setminus \{1/2, 1\}$ we compute $J(\mu_1, 1-\mu_1)$ numerically using Gaussian quadratures. In Table 1 and in Figure 1 we give values of $\mathcal{E}(\mu_1)$ for $0.5 \leq \mu_1 := m_1/(m_1+m_2) \leq 1$.

4 Conclusions

In this paper we give a rigorous computation, with error bound, of the entanglement created in the low-energy scattering of two particles in two dimensions. The interaction between the particles is given by potentials that are not required to be spherically symmetric. Before the scattering the particles are in a pure state that is a product of two normalized Gaussians with the same variance σ. After the collision the particles are in a outgoing asymptotic state that is not a product state. The measure of the entanglement created by the collision is the purity, $\mathcal{P}$, of one of the particles in the state after the collision. Before the collision the purity is one.

We prove that $\mathcal{P} = 1 - \frac{1}{(\ln(\sigma/\hbar))^2}\, \mathcal{E} + O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right)$, as $\sigma/\hbar \to 0$, where σ is the variance of the and the entanglement coefficient, $\mathcal{E}$, depends only on the masses of the particles and not on the interaction potential. This proves that the entanglement created at low-energy in two dimensions is universal, in the sense that it is independent on the interaction potential between the particles. This is strikingly different with the three dimensional case, that we considered in [3], where the entanglement created at low-energy is proportional to the total scattering cross section. However, the entanglement depends strongly in the difference of the masses. As in three dimensions [3] it takes its minimum when the masses are equal, and it increases rapidly with the difference of the masses.

5 Appendix

By (3.71) we have that [2]

$$L(\mu_1, 1-\mu_1) = \int_0^\infty d\lambda\, e^{-2\lambda}\, \frac{2}{1+(2\mu_1-1)^2}\, I_0\left(\frac{(2\mu_1-1)^2\lambda}{1+(2\mu_1-1)^2}\right) \mathrm{Exp}\left[\frac{(2\mu_1-1)^2\lambda}{1+(2\mu_1-1)^2}\right] = \frac{1}{\sqrt{1+(2\mu_1-1)^2}}. \tag{5.1}$$

Moreover, by (3.70) and denoting $\mathbf{q}_{\rm cm} := \mathbf{q}_1 + \mathbf{q}_2$,

$$J(1,0) = \int d\mathbf{q}_2\, e^{-3\mathbf{q}_2^2} \left[\int d\mathbf{q}_{\rm cm}\, \mathrm{Exp}\left(-\mathbf{q}_{\rm cm}^2 + \mathbf{q}_{\rm cm}\cdot\mathbf{q}_2\right) I_0(|\mathbf{q}_{\rm cm}||q_2|)\right]^2. \tag{5.2}$$

Furthermore, using polar coordinates, [2],

$$\begin{aligned} J(1,0) &= \pi^3 \int_0^\infty d\lambda\, e^{-3\lambda} \left[\int_0^\infty d\rho\, e^{-\rho} \left(I_0(\sqrt{\rho}\sqrt{\lambda})\right)^2\right]^2 = \\ &\pi^3 \int_0^\infty d\lambda\, e^{-2\lambda} \left(I_0(\lambda/2)\right)^2 = \pi^2 K(0.25) = 16.6377, \end{aligned} \tag{5.3}$$

where $K(x)$ is the complete elliptic integral.

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Table 1: The Entanglement Coefficient $\mathcal{E}(\mu_1)$

$\mu_1 := m_1/(m_1 + m_2)$	$\mathcal{E}(\mu_1)$
0.5	0.000
0.525	0.0001
0.55	0.0012
0.575	0.0057
0.6	0.0174
0.625	0.0408
0.65	0.0806
0.675	0.1410
0.7	0.2253
0.725	0.3357
0.75	0.4725
0.775	0.6348
0.8	0.8203
0.825	1.0255
0.85	1.2462
0.875	1.4776
0.9	1.7151
0.925	1.9542
0.95	2.1909
0.975	2.4216
1	2.6436

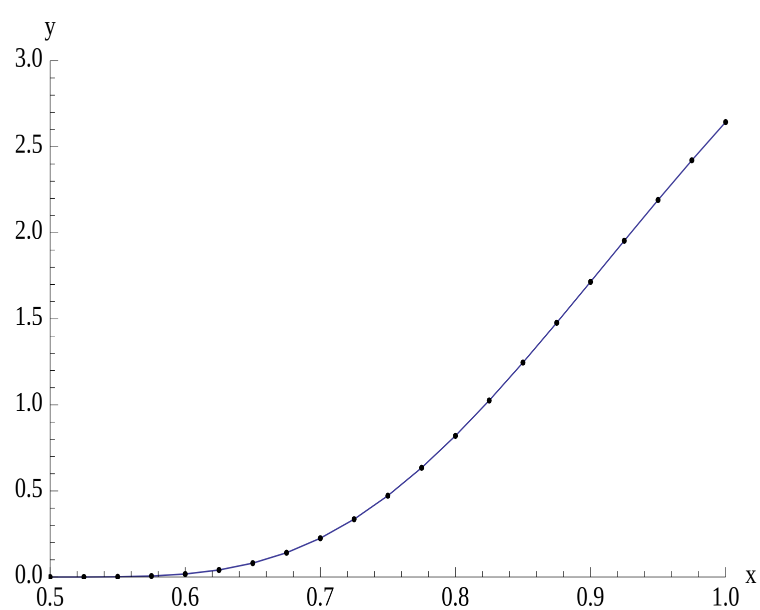

Figure 1: The entanglement coefficient $y = \mathcal{E}(\mu_1)$, as a function of $x = \mu_1 = m_1/(m_1 + m_2)$, for $0.5 \leq \mu_1 \leq 1$.